ENERGY-MOMENTUM TENSORS IN GAUGE THEORY

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Summary
In field theory on a fibre bundle $Y \to X$, an energy-momentum current is associated to a lift onto Y of a vector field on X. Such a lift by no means is unique and contains a vertical part. It follows that: (i) there are a set of different energy-momentum currents; (ii) the Noether part of an energy-momentum current is not taken away; (iii) if a Lagrangian is not gauge-invariant, the energy-momentum fails to be conserved.

In gauge theory, classical fields are represented by sections of a fibre bundle $Y \to X$, coordinated by (x^λ, y^i). Their configuration space is the first order jet manifold J^1Y of $Y \to X$ coordinated by $(x^\lambda, y^i, y^i_\lambda)$ (where y^i_λ are coordinates of derivatives of field functions). A Lagrangian on J^1Y is defined as a density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)d^nx, \qquad n = \dim X.$$

By gauge transformations are meant bundle automorphisms of $Y \to X$. To study Lagrangian conservation laws, it suffices to consider 1-parameter groups G_u of gauge transformations. Their infinitesimal generators are projectable vector fields $u = u^\lambda(x)\partial_\lambda + u^i(y)\partial_i$ on $Y \to X$. A Lagrangian L is G_u-invariant iff its Lie derivative $\mathbf{L}_uL$ along u vanishes. The first variational formula states the canonical decomposition

$$\mathbf{L}_uL = (u^i - y^i_\mu u^\mu)\mathcal{E}_i d^nx - d_\lambda \mathfrak{T}^\lambda_u d^nx, \qquad d_\lambda = \partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial^\mu_i, \tag{1}$$

where $\mathcal{E}_i = (\partial_i\mathcal{L} - d_\lambda\partial^\lambda_i\mathcal{L})$ is the Euler–Lagrange operator and

$$\mathfrak{T}^\lambda_u = (u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L} \tag{2}$$

is the current along u. On the shell $\mathcal{E}_i = 0$, the first variational formula (1) leads to the weak identity

$$\mathbf{L}_uL \approx -d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}].$$

If the Lie derivative $\mathbf{L}_uL$ vanishes, we obtain the weak conservation law

$$0 \approx -d_\lambda\mathfrak{T}^\lambda_u$$

of the current $\mathfrak{T}_u$ (2).

Remark 1. It may happen that a current $\mathfrak{T}$ (2) takes the form

$$\mathfrak{T}^\lambda = W^\lambda + d_\mu U^{\mu\lambda},$$

where the term W vanishes on-shell ($W \approx 0$). Then one says that $\mathfrak{T}$ reduces to a superpotential U.

Remark 2. Background fields do not live in the dynamic shell $\mathcal{E}_i = 0$ and, therefore, break Lagrangian conservation laws as follows. Let us consider the product $Y_{\rm tot} = Y \times Y'$ of the above fibre bundle Y of dynamic fields and a fibre bundle Y', coordinated by (x^λ, y^A), whose sections are background fields. A Lagrangian L is defined on the total configuration space $J^1Y_{\rm tot}$. Let

$$u = u^\lambda(x)\partial_\lambda + u^A(x^\mu, y^B)\partial_A + u^i(x^\mu, y^B, y^j)\partial_i \tag{3}$$

be a projectable vector field on $Y_{\rm tot}$ which also projects onto Y' because gauge transformations of background fields do not depend on the dynamic ones. Substitution of (3) in (1) leads to the first variational formula in the presence of background fields

$$\begin{aligned}\mathbf{L}_u L = (u^A - y^A_\lambda u^\lambda)\partial_A\mathcal{L} + \partial^\lambda_A\mathcal{L} d_\lambda(u^A - y^A_\mu u^\mu) +\\ (u^i - y^i_\lambda u^\lambda)\mathcal{E}_i - d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}].\end{aligned}$$

A total Lagrangian L is usually invariant under gauge transformations of the product $Y \times Y'$. In this case, we obtain the weak identity

$$0 \approx (u^A - y^A_\lambda u^\lambda)\mathcal{E}_A - d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}] \tag{4}$$

in the presence of background on the dynamic shell $\mathcal{E}_i = 0$.

Point out the following properties of currents.

(i) $\mathfrak{T}_{u+u'} = \mathfrak{T}_u + \mathfrak{T}_{u'}$.

(ii) Any projectable vector field u on Y projected onto the vector field $\tau = u^\lambda\partial_\lambda$ on X is written as the sum $u = \widetilde{\tau} + u_V$ of some lift $\widetilde{\tau}$ of τ onto Y and the vertical vector field $u_V = u - \widetilde{\tau}$ on Y.

(iii) The current along a vertical vector field $u = u^i\partial_i$ on Y is the Noether current $\mathfrak{T}^\lambda_u = -u^i\partial^\lambda_i\mathcal{L}$.

(iv) The current $\mathfrak{T}_{\widetilde{\tau}}$ along a lift $\widetilde{\tau}$ onto Y of a vector field $\tau = \tau^\lambda\partial_\lambda$ on X is said to be the energy-momentum current.

It follows from the items (i) – (iv) that any current can be represented by a sum of an energy-momentum current and a Noether one.

Different lifts $\widetilde{\tau}$ and $\widetilde{\tau}'$ onto Y of a vector field τ on X lead to distinct energy-momentum currents $\mathfrak{T}_{\widetilde{\tau}}$ and $\mathfrak{T}_{\widetilde{\tau}'}$, whose difference $\mathfrak{T}_{\widetilde{\tau}} - \mathfrak{T}_{\widetilde{\tau}'}$ is the Noether current along the vertical vector field $\widetilde{\tau} - \widetilde{\tau}'$ on Y. The problem is that, in general, there is no canonical lift onto Y of vector fields on X, and one can not take the Noether part away from an energy-momentum current.

There exists the category of so called natural bundles $T \to X$, exemplified by tensor bundles, which admit the canonical lift $\widetilde{\tau}$ onto T of any vector field τ on X. This is the case of space-time symmetries and gravitation theory. Such a lift is the infinitesimal generator of a 1-parameter group of general covariant transformations of T. The corresponding energy-momentum current $\mathfrak{T}_{\widetilde{\tau}}$ is reduced to the generalized Komar superpotential. Other energy-momentum currents differ from $\mathfrak{T}_{\widetilde{\tau}}$ in the Noether ones, but they fail to be conserved because almost all gravitation Lagrangians are not invariant under vertical (non-holonomic) gauge transformations.

Let us focus on field models on non-natural bundles $Y \to X$, i.e., they possess internal symmetries. Then a vector field on X gives rise to Y by means of a connection on $Y \to X$.

A connection on a fibre bundle $Y \to X$ is defined as a section Γ of the affine jet bundle $J^1Y \to Y$, and is represented by the tangent-valued form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{5}$$

It follows that connections on $Y \to X$ make up an affine space modelled on the space of soldering forms $\sigma = \sigma^i_\lambda dx^\lambda \otimes \partial_i$. In particular, the difference of two connections $\Gamma - \Gamma'$ is a soldering form. Given a connection Γ (5), a vector field $\tau = \tau^\lambda \partial_\lambda$ on X gives rise to the projectable vector field

$$\Gamma\tau = \tau^\lambda(\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

on a fibre bundle Y. The corresponding current

$$\mathfrak{T}^\lambda_{\Gamma\tau} = \tau^\mu T^\lambda_\mu = \tau^\mu[(y^i_\mu - \Gamma^i_\mu)\partial^\lambda_i \mathcal{L} - \delta^\lambda_\mu \mathcal{L}]$$

is called the energy-momentum current, while T^λ_μ is said to be the energy-momentum tensor with respect to a connection Γ.

Remark 3. The difference $\mathfrak{T}_{\Gamma\tau} - \mathfrak{T}_{\Gamma'\tau}$ of energy momentum currents with respect to different connections Γ and Γ' is the Noether current along the vertical vector field $\Gamma\tau - \Gamma'\tau = \tau^\lambda(\Gamma^i_\lambda - \Gamma'^i_\lambda)\partial_i$.

Remark 4. Let $Y \to X$ be a trivial bundle and Γ a flat connection on it. There is a coordinate system on Y such that $\Gamma^i_\lambda = 0$. Then the energy-momentum tensor with respect to Γ reduces to the familiar canonical energy-momentum tensor. The latter, however, is not preserved under gauge transformations, and it is not defined on a non-trivial bundle.

Let us study energy-momentum conservation laws in gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group G. These connections are sections of the fibre bundle $C = J^1P/G \to X$, coordinated by (x^λ, a^q_λ), and are identified with gauge potentials. Their configuration space is the jet manifold J^1C coordinated by $(x^\lambda, a^q_\lambda, a^q_{\lambda\mu})$. It admits the canonical splitting

$$a^r_{\lambda\mu} = \frac{1}{2}(\mathcal{F}^r_{\lambda\mu} + \mathcal{S}^r_{\lambda\mu}) = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq} a^p_\lambda a^q_\mu) + \frac{1}{2}(a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu).$$

Gauge transformation in gauge theory on a principal bundle $P \to X$ are automorphism of $P \to X$ which are equivariant under the canonical action of the structure group G on P on the right. They induce the automorphisms of the bundle of connections C whose generators read

$$\xi = \xi^\lambda \partial_\lambda + (\partial_\mu \xi^r + c^r_{pq} a^p_\mu \xi^q - a^r_\lambda \partial_\mu \xi^\lambda) \partial^\mu_r, \tag{6}$$

where ξ^r are functions on X which play the role of gauge parameters.

Let L be a Lagrangian on J^1C. One usually requires of L to be invariant under vertical gauge transformations with the generators

$$\xi = (\partial_\lambda \xi^r + c^r_{qp} a^q_\lambda \xi^p) \partial^\lambda_r.$$

Hence, L is a function of the strength $\mathcal{F}$. Then the Noether current

$$\mathfrak{T}^\lambda_\xi = -(\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p) \partial^{\lambda\mu}_r \mathcal{L}$$

is conserved. It reduces to the superpotential form

$$\mathfrak{T}^\lambda_\xi = \xi^r \mathcal{E}^\lambda_r + d_\mu U^{\mu\lambda}, \qquad U^{\mu\lambda} = \xi^p \partial^{\lambda\mu}_p \mathcal{L}.$$

Given a principal connection B on $P \to X$, there exists the lift

$$\widetilde{\tau}_B = \tau^\lambda \partial_\lambda + [\partial_\mu(\tau^\lambda B^r_\lambda) + c^r_{qp} a^q_\mu (\tau^\lambda B^p_\lambda) - a^r_\lambda \partial_\mu \tau^\lambda] \partial^\mu_r. \tag{7}$$

of a vector field τ on X onto the bundle of connections $C \to X$. It is a generator (6) of gauge transformations of C with the gauge parameters $\xi^r = \tau^\lambda B^r_\lambda$.

Discovering the energy-momentum current along the lift (7), we assume that a Lagrangian L of gauge theory depends on a background metric on X. This metric is a section of the tensor bundle $\overset{2}{\vee} TX$ coordinated by $(x^\lambda, \sigma^{\mu\nu})$. Following Remark 2, we define L on the total configuration space $J^1Y = J^1(C \underset{X}{\times} \overset{2}{\vee} TX)$. Given a vector field τ on X, there exists its canonical lift

$$\widetilde{\tau}_g = \tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha}) \partial_{\alpha\beta} \tag{8}$$

onto the tensor bundle $\overset{2}{\vee} TX$. Combining (7) and (8) gives the lift

$$\widetilde{\tau}_Y = [\tau_g - a^r_\lambda \partial_\mu \tau^\lambda \partial^\mu_r] + [\partial_\mu(\tau^\lambda B^r_\lambda) + c^r_{qp} a^q_\mu (\tau^\lambda B^p_\lambda)] \partial^\mu_r$$

of a vector field τ on X onto the product Y. The first term in this expression is a local generator of general covariant transformations, while the second one is that of vertical gauge transformations.

Let a total Lagrangian L be invariant under general covariant transformations and vertical gauge transformations. Then using the formula (4), we obtain the weak identity

$$0 \approx \partial_\lambda \tau^\mu t^\lambda_\mu \sqrt{\mid g \mid} - \tau^\mu \{_\mu{}^\beta{}_\lambda\} t^\lambda_\beta \sqrt{\mid g \mid} - d_\lambda \mathfrak{T}^\lambda_B, \tag{9}$$

where t^λ_μ is the metric energy-momentum tensor, $\{_\mu{}^\beta{}_\lambda\}$ are the Christoffel symbols of a background metric g, and

$$\mathfrak{T}^\lambda_B = [\partial^{\lambda\nu}_r \mathcal{L}(\tau^\mu a^r_{\mu\nu} + \partial_\nu \tau^\mu a^r_\mu) - \tau^\lambda \mathcal{L}] + [-\partial^{\lambda\nu}_r \mathcal{L}(\partial_\nu(\tau^\mu B^r_\mu) + c^r_{qp} a^q_\nu (\tau^\mu B^p_\mu)] \tag{10}$$

is the energy-momentum current along the vector field (7). If L is the Yang–Mills Lagrangian, a simple computation brings (9) into the familiar covariant conservation law

$$\nabla_\lambda (t^\lambda_\mu \sqrt{\mid g \mid}) \approx 0, \tag{11}$$

independent of a connection B. All other energy-momentum conservation laws differ from (11) in a superpotential term $d_\mu d_\lambda U^{\mu\lambda}$.

If a Lagrangian L is not gauge-invariant, no energy-momentum is conserved. For instance, let

$$L = \frac{1}{2k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} a^m_\alpha (\mathcal{F}^n_{\lambda\mu} - \frac{1}{3} c^n_{pq} a^p_\lambda a^q_\mu) d^3x$$

be the Chern–Simons Lagrangian. It is not gauge-invariant, but its Euler–Lagrange operator is so. Then we obtain the conservation law

$$0 \approx -d_\lambda [\mathfrak{T}^\lambda_B + \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \partial_\alpha (\tau^\nu B^m_\nu) a^n_\mu],$$

where $\mathfrak{T}_B$ is the energy-momentum current (10) along the vector field $\widetilde{\tau}_B$. Thus, the energy-momentum current of the Chern–Simons model is not conserved, but there exists another conserved quantity.

Another interesting example is a Lagrangian in the generating functional of quantum gauge theory. It is not gauge-invariant, but is BRST-invariant. The corresponding energy-momentum current is conserved, but it contains ghost fields.

References

G.Sardanashvily, Energy-momentum conservation laws in gauge theory with broken invariance, *E-print arXiv*: **hep-th/0203275**

L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

G.Sardanashvily, Stress-energy-momentum tensors in constraint field theories, *J. Math. Phys.* **38** (1997) 847.

G.Sardanashvily, Stress-energy-momentum conservation law in gauge gravitation theory, *Class. Quant. Grav.* **14** (1997) 1371.